HEF-003-1501001 Seat No. ______

M. Phil. (Science / Maths) (Sem. I) Examination

December – 2017

Mathematics : CMT-10001

(Algebra)

(New Course)

Faculty Code : 003
Subject Code : 1501001

Time : **3** Hours] [Total Marks : **100**

Instructions : (1) Answer all the questions.
(2) Each question carries 20 marks.

1 Answer any **ten** : **10×2=20**

(a) Give an example of a commutative ring which does not contain unit element.

(b) Define unit in a ring and field.

(c) Write down at least two facts of an ideal I in a ring R.

(d) Let $f: R \to S$ be a ring homomorphism and $\ker f = \{r \in R / f(r) = 0\}$. Prove that f is $1-1 \Leftrightarrow \ker f = (0)$.

(e) Define zero divisor of a ring R and write down at least five zero divisors of ring Z_{18}.

(f) Define local ring and semilocal ring.

(g) Define multiplicative closed set in a ring R. For a ring R with $1 \in R$, prove that $\{1, x, x^2, ...\}$ is an mc set of R, where $x \in R$.

(h) Define nilradical and Jacobson radical.

(i) In standard notation prove that $J(z) = (0)$.

(j) Let M_1, M_2 be two distinct maximal ideals of a ring *R*. Prove that $M_1 + M_2 = R$.

(k) Define $\sqrt{I}$, radical of an ideal *I* in *R*. Write down at least two properties of radical of ideals.

(l) In standard notation prove that

$$(I_1 + I_2)^e = I_1^e + I_2^e$$

(m) Define Noetherian ring and Artinian ring.

(n) Define an exact sequence of *R*-modules and a short exact sequence of *R*-modules.

2 Answer any **four** : **4×5=20**

(a) Let $\{P_\alpha\}_{\alpha \in \wedge}$ be a collection of prime ideals of a ring *R*. Prove that $R - \left(\bigcup_{\alpha \in \wedge} p_\alpha \right)$ is an mc set of *R*.

(b) Let *R* be a ring. In standard notation prove that $J(R) = \{x \in R / 1 - rx \text{ is a unit in } R, \ \forall\, r \in R\}$.

(c) Let $I_1, I_2, \ldots\ldots.., I_n$ be distinct ideals of *R* such that $I_j + I_k = R, \quad \forall, j, k, \in \{1, 2, \ldots\ldots.., n\}$ and $j \neq k$, Prove that $\prod_{j=1}^{n} I_j = \bigcap_{k=1}^{n} I_k$.

(d) Prove that $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

(e) Prove that only Artinian integral domain is a field.

(f) Prove that homomorphic image of a Noetherian *R*-module is also Noetherian.

(g) In standard notation prove that $s^{-1}\left(\sqrt{I}\right) = \sqrt{s^{-1} I}$.

3 Answer any **one** : **1×20=20**

(a) Let R be a Noetherian ring. Prove that $R[x]$ is also Noetherian.

(b) State and prove (i) Nakayama's Lemma (ii) Chinese Remainder Theorem.

(c) Let S be an mc subset of a ring R and $g : R \to T$ be a ring homomorphism. Let $g(s)$ be a unit in $T, \forall s \in S$. Prove that $\exists$ a ring homomorphism $h : S^{-1}R \to T \ni$ $hof = g$, where $f : R \to S^{-1}R$ and $f(r) = \frac{r}{1}$. Also prove uniqueness of h.

4 Answer any **two** : **2×10=20**

(a) Let $R = C_{\mathbb{R}}[0, 1]$. Prove that $M_{t_0} = \{f \in R / f(t_0) = 0\}$ is a maximal ideal of R.

(b) Let M_1, M_2 be R-submodules of an R-modules M. Prove that

(i) $M_1 + M_2$ is an R-submodule of M.

(ii) $\frac{M_1 + M_2}{M_1} \simeq \frac{M_2}{M_1 \cap M_2}$ as R-modules.

(c) Define cofinitely generated R-module. Prove that an R-module M satisfies dcc iff $\frac{M}{N}$ is a cofinitely generated R-module, for any R-submodule N of M.

(d) Let R be an Artinian ring. Prove that nil(R) is a nilpotent ideal of R.

(e) Let R be a ring and S be an mc subset of R. Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact sequence of R-modules. Prove that $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is also an exact sequence of S^{-1} R-modules.

5 Answer any **five** : **5×4=20**

(1) Let $p_1, p_2, ..., p_{10}$ be distinct primes. Prove that $\dfrac{\mathbb{Z}}{\prod_{i=1}^{10} p_i\mathbb{Z}}$ has no non-zero nilpotent element. Also write down two zero divisors of the ring $\dfrac{\mathbb{Z}}{\prod_{i=1}^{10} p_i\mathbb{Z}}$.

(2) Prove that $\sqrt{\bigcap_{i=1}^{n} I_i} = \sqrt{\prod_{i=1}^{n} I_i} = \bigcap_{i=1}^{n} \sqrt{I_i}$.

(3) Let $p_1, p_2, p_3, ..., p_n$ be prime ideals of a ring R and I be an ideal of R such that $I \subseteq \bigcup_{i=1}^{n} p_i$. Prove that $I \subseteq p_j$, for some $j \in \{1, 2, ..., n\}$.

(4) In standard notation prove that (1) $I \subseteq I^{ec}$ and (2) $J^{ce} \subseteq J$, where I is an ideal of R, J is an ideal of T and $f : R \to T$ is a ring homomorphism.

(5) Let $f : M \to N$ be an R-homomorphism of R-modules M, N. Prove that (1) $\{m \in M / f(m) = 0\}$ is an R-submodule of M and (2) $\{f(m) / m \in M\}$ is an R-submodule of N.

(6) Let M be an R-module and N be an R-submodule of M. Let N and M/N both are *f.g.* R-modules. Prove that M is also a *f.g.* R-module.

(7) Let R be a *PID* and P be a non-zero prime ideal of $R(P \neq (0))$. Prove that P is a maximal ideal of R.